

A quantum ergodic theorem for mapping class groups action on character variety

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Abstract

We state a theorem relating the ergodicity of the action of a given subgroup of the mapping class group of a surface on the character variety, to the asymptotic of its invariant subspaces through the Reshetikhin-Turaev representations. As application we give an asymptotic result on the spin decomposition arising in [BHMV95]

Keywords: Reshetikhin-Turaev representations, mapping class group, character variety, quantum chaos.

1 Introduction

Witten gave in [Wit89] convincing arguments for the existence of Topological Field Theories, as defined in [Ati88, Wit88], giving a three dimensional interpretation of the Jones polynomial when the gauge group is $SU(2)$. Soon after, Reshetikhin and Turaev gave a rigorous construction of these TQFTs [RT91] using a modular category based on the representations of certain quantum groups. We can extract from each of these TQFT a projective unitary representation of the mapping class group $\text{Mod}(\Sigma_g)$ of a genus g closed oriented surface Σ_g .

Even though our results work in a more general context, we will focus on the representations associated to the gauge group $SU(2)$, that we will denote by:

$$\rho_p : \text{Mod}(\Sigma_g) \rightarrow \text{PU}(V_p(\Sigma_g)) \quad , \text{ for } p \geq 3$$

where $V_p(\Sigma_g)$ is a finite dimensional complex vector space and $p \geq 3$ an integer, called level, indexing the representations.

These representations are closely related to the action of $\text{Mod}(\Sigma_g)$ on the character variety:

$$\mathcal{X}_g := \text{Hom}(\pi_1(\Sigma_g), SU(2)) / SU(2)$$

equipped with its natural symplectic form ω .

The spaces $\text{End}(V_p(\Sigma_g))$ can be thought as quantizations of the commutative algebra $\mathcal{O}_{\mathcal{X}_g}$ of regular functions on \mathcal{X}_g . We now describe the relation between these algebras that we will need in this paper.

Denote by \mathcal{M}_g the set of isotopy classes of multicurves in Σ_g , that is of one dimensional submanifolds of Σ_g (including the empty one), with no contractible component.

Given $\gamma \in \mathcal{M}_g$, we can define a regular function $f_\gamma \in \mathcal{O}_{\mathcal{X}_g}$ by the formula:

$$f_\gamma([\rho]) = -\text{Tr}(\rho(\gamma)) \quad , \text{ for } [\rho] \in \mathcal{X}_g$$

It is shown in [Bul97, CM09] that the f_γ form a vectorial basis of $\mathcal{O}_{\mathcal{X}_g}$. However, the algebra $\mathcal{O}_{\mathcal{X}_g}$ is equipped with a natural tracial state τ given by:

$$\tau(f) := \frac{1}{\text{Vol}(\mathcal{X}_g)} \int_{\mathcal{X}_g} f dV$$

where dV is the measure associated to the symplectic form ω and $\text{Vol}(\mathcal{X}_g)$ is the symplectic volume.

On the other hand, we can also associate to $\gamma \in \mathcal{M}_g$ a (curve) operator $\text{Add}_p(\gamma) \in \text{End}(V_p(\Sigma_g))$, which we be defined in the next section, such that the set of curve operators is a generating set of $\text{End}(V_p(\Sigma_g))$. This algebra has a natural tracial state B_p defined by:

$$B_p(M) := \frac{1}{\dim(V_p(\Sigma_g))} \text{Tr}(M)$$

In [MN08], the authors proved the following relation between these tracial algebras which is the key property of this paper:

Marché - Narrimanejad [MN08] For all $\gamma \in \mathcal{M}_g$, we have:

$$B_p(\text{Add}_p(\gamma)) \xrightarrow{p \rightarrow \infty} \tau(f_\gamma) \quad (1)$$

We now state the main theorem of this paper:

Theorem 1.1.

Let G be a subgroup of $\text{Mod}(\Sigma_g)$ acting ergodically on \mathcal{X}_g .

For each $p \geq 3$, let

$$V_p(\Sigma_g) = W_{1,p} \oplus \dots \oplus W_{N_p,p}$$

be a decomposition of $V_p(\Sigma_g)$ into G -invariant subspaces.

Denote by $P_{i,p}$ the orthogonal projection on $W_{i,p}$.

Then there exist sets $J_p \subset \{1, \dots, N_p\}$ such that:

1. *We have:*

$$\frac{1}{\dim V_p(\Sigma_g)} \sum_{i \in J_p} \dim(W_{i,p}) \xrightarrow{p \rightarrow \infty} 1$$

2. *For any sequence $j = (j_p)_{p \geq 3}$ with $j_p \in J_p$, we have:*

$$B_p(P_{j_p,p} \text{Add}_p(\gamma)) \xrightarrow{p \rightarrow \infty} \tau(f_\gamma)$$

In section 2 we will state a more general version of this theorem that we wish to apply to the geometric quantization of general character variety $\text{Hom}(\pi_1(\Sigma_g), H) / H$, where H is a compact, simply laced Lie group, or the group $U(1)$.

When the gauge group H is $U(1)$, the associated representations are the so-called Weil representations and the phase space is the symplectic torus \mathbb{T}^g . When the genus is one and $G = \mathbb{Z}$ whose generator acts on the torus by an Anosov elements, then we recover the Schnirelman-type theorem of Bouzouina and De Bièvre of [BDB96]. Our theorem is thus a generalization of this quantum ergodic theorem when both G and H are non-abelian.

To apply this theorem, we need groups acting ergodically on \mathcal{X}_g for the symplectic measure. In [Gol97], Goldman showed that the action of $\text{Mod}(\Sigma_g)$ is ergodic. In [FM13], Funar and Marché showed that the action of the first Johnson subgroup of $\text{Mod}(\Sigma_g)$ is also ergodic. In [BHMV95], non trivial invariant subspaces for both $\text{Mod}(\Sigma_g)$ and the Torelli group were found when 4 divides p .

Denote by \mathcal{H} the group:

$$\mathcal{H} := \begin{cases} H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}) & , \text{ if } p \equiv 4 \pmod{8} \\ H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z} & , \text{ if } p \equiv 0 \pmod{8} \end{cases}$$

where the semi-direct product is the only non trivial one. In [BHMV95], the authors defined a non trivial decomposition:

$$V_p(\Sigma_g) = \bigoplus_{\chi \in \text{Hom}(\mathcal{H}, \mathbb{Z}/2\mathbb{Z})} V_p(\Sigma_g, \chi)$$

where each $V_p(\Sigma_g, \chi)$ is invariant under the Torelli group and $V_p(\Sigma_g, \chi = 0)$ is invariant under $\text{Mod}(\Sigma_g)$. We can deduce from Theorem 1.1 the following:

Corollary 1.2. *Let $(p_n)_n$ be an increasing sequence of non-negative integers, all of which being congruent to either 4 or 0 modulo 8, and let $\chi \in \text{Hom}(\mathcal{H}, \mathbb{Z}/2\mathbb{Z})$. Denote by P_χ the orthogonal projector on $V_p(\Sigma_g, \chi)$. Then we have:*

$$B_{p_n}(P_\chi \text{Add}_{p_n}(\gamma)) \xrightarrow[n \rightarrow \infty]{} \tau(f_\gamma) \quad , \text{ for all } \gamma \in \mathcal{M}_g$$

The paper is organized as follows. In section 2 we briefly define the Reshetikhin-Turaev representations using skein theory following [Lic91, BHMV95]. Nothing new is claimed here. Section 3 is devoted to the statement and the proof of the main theorem. In the last section we prove Corollary 1.2.

Acknowledgements: The author is thankful to L.Charles, L.Funari, J.Marché and F.Paulin for useful discussions. He also thanks C.Oliveira, A.E.Presotto, F.Ruffino, D.Vendruscolo and the mathematic department of UFSCar for their kind hospitality during the redaction of this paper. He acknowledge support from the grant ANR 2011 BS 0102001 ModGroup, the GDR Tresses, the GDR Platon and the GEAR Network.

2 Skein construction of the Reshetikhin-Turaev representations

Following [BHMV95], we will briefly define the representations ρ_p and fix some notations.

2.1 The spaces $V_p(\Sigma_g)$

Given an even integer $p \geq 6$, we note $A := -\exp(i\pi/p) \in \mathbb{C}$, a primitive $2p$ -th root of unity. Using the Kauffman skein relation of Figure 1, we associate to any framed link $L \subset S^3$ an invariant $\langle L \rangle_p \in \mathbb{C}$.

$$\begin{aligned} \text{Crossing} &= A \left(\text{Positive Twist} - \text{Negative Twist} \right) \\ \text{Single Component} &= -(A^2 + A^{-2}) \emptyset \end{aligned}$$

Figure 1: Skein relations defining the framed link invariants.

Choose $g \geq 1$ and denote by C_g the set of isotopy classes of framed links (including the empty link) in an oriented genus g handlebody H_g . We fix a genus g Heegaard splitting of the sphere, i.e.

an element $S \in \text{Mod}(\Sigma_g)$ and two handlebodies so that :

$$H_g^1 \bigcup_{S: \partial H_g^1 \rightarrow \partial H_g^2} H_g^2 \cong S^3$$

Take $L_1, L_2 \in C_g$ and embed L_1 in H_g^1 and L_2 in H_g^2 . The above gluing defines a link $L_1 \bigcup_S L_2 \subset S^3$. We call Hopf pairing the bilinear form:

$$(\cdot, \cdot)_{g,p}^H : \mathbb{C}[C_g] \times \mathbb{C}[C_g] \rightarrow \mathbb{C}$$

defined by

$$(L_1, L_2)_{p,g}^H := \left\langle L_1 \bigcup_S L_2 \right\rangle_p$$

Eventually we define the spaces $V_p(\Sigma_g)$ as the quotients:

$$V_p(\Sigma_g) := \mathbb{C}[C_g] / \ker \left((\cdot, \cdot)_{g,p}^H \right)$$

The vector spaces $V_p(\Sigma_g)$ are finite dimensional ([BHMV95]).

2.2 The invariant form $\langle \cdot, \cdot \rangle_p$

We now define a definite positive form $\langle \cdot, \cdot \rangle_p : V_p(\Sigma_g)^{\otimes 2} \rightarrow \mathbb{C}$ which will be invariant for the action of $\text{Mod}(\Sigma_g)$. Using the identification $H_g \cong D_g^2 \times [0, 1]$, where D_g^2 denotes the g -holed disc, we define a 'stack product' $H_g \times H_g \cong H_g$ by using the homeomorphism $[0, 1] \times [0, 1] \cong [0, 1]$ given by $(x, y) \rightarrow \frac{1}{2}(x + y)$.

Now when identifying $D_g^2 \times 0$ with $D_g^2 \times 1$ in H_g using the identity map, we get g connected sums $(D^2 \times S^1) \# \dots \# (D^2 \times S^1)$. Composing with the preceding stack product, we get a continuous map:

$$T : H_g \times H_g \rightarrow (D^2 \times S^1) \# \dots \# (D^2 \times S^1)$$

Denote by K_g the set of isotopy classes of framed links in $(D^2 \times S^1) \# \dots \# (D^2 \times S^1)$. It results from classical surgery properties of reduced skein modules, that the space $\mathbb{C}[K_g] / \sim$, obtained by quotienting the \mathbb{C} vector space generated by elements of K_g by the skein relations of Figure 1, is one dimensional. We thus have an isomorphism $\mathbb{C}[K_g] / \sim \cong \mathbb{C}$ which sends the empty link to 1.

Now the map:

$$\tilde{T} : C_g \times C_g \rightarrow K_g$$

sending (L_1, L_2) to $T(L_1, L_2)$, induces by linearity a form:

$$\langle \cdot, \cdot \rangle_p : V_p(\Sigma_g) \times V_p(\Sigma_g) \xrightarrow{\tilde{T}} K_g \cong \mathbb{C}$$

In [BHMV95], the authors exhibit an orthogonal basis for this form and proved it is definite positive.

2.3 The Reshetikhin-Turaev representations

Denote by $x_1, \dots, x_g, y_1, \dots, y_g, z_1, \dots, z_{g-1}$ the oriented curves of Σ_g drawn in Figure 2. In [Hum77], Humphries showed that their associated Dehn twists generate $\text{Mod}(\Sigma_g)$.

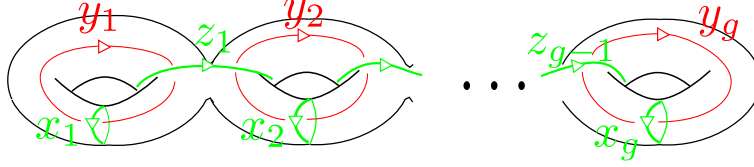


Figure 2: Humphries' generators of $\text{Mod}(\Sigma_g)$.

We fix an orientation preserving homeomorphism

$$\alpha : \Sigma_g \rightarrow \partial H_g$$

such that the x_i 's and z_i 's are contractible in H_g through α and the y_i 's are contractible through $\alpha \circ S$.

If $\phi \in \text{Mod}(\Sigma_g)$ is a class associated to a homeomorphism which extends to H_g through α , such as the T_{x_i} 's and the T_{z_i} 's, then ϕ acts on C_g and preserves the kernel of the Hopf pairing so acts on $V_p(\Sigma_g)$ by passing to the quotient. Denote by $\rho_p(\phi) \in \text{GL}(V_p(\Sigma_g))$ the resulting operator.

Now choose $\phi \in \text{Mod}(\Sigma_g)$ such that the corresponding homeomorphisms extend to H_g through $\alpha \circ S$, such as the T_{y_i} 's. This extension also defines, by quotient, an operator on $V_p(\Sigma_g)$. We denote by $\rho_p(\phi)$ the dual of this operator for the Hopf pairing. Figure 3 illustrates these operators when $g = 1$.

$$\begin{aligned} & \text{Diagram of } T, T' \text{ Dehn Twists} \\ & \rho(T) \left(\text{Diagram of } T \right) = \text{Diagram of } T \text{ with a blue loop} = -A^{-3} \left(\text{Diagram of } T \right) \\ & \langle \text{Diagram of } T, \rho(T') \left(\text{Diagram of } T' \right) \rangle = \langle \rho(T) \left(\text{Diagram of } T \right), \text{Diagram of } T' \rangle \\ & = \langle \text{Diagram of } T \text{ with a red loop}, \text{Diagram of } T' \rangle_p = A^{11} + 2A^7 + 2A^3 + 2A^{-1} + A^{-5} \end{aligned}$$

Figure 3: Action of the Humphries' generators on $V_p(\Sigma_1)$.

We will now show that the classes of the operators $\rho_p(\phi)$ generate a projective representation:

$$\rho_p : \text{Mod}(\Sigma_g) \rightarrow \text{PGL}(V_p(\Sigma_g))$$

Since each of the preceding generating operators preserve $\langle \cdot, \cdot \rangle_p$, this representation is however unitary.

Let $\gamma \in \mathcal{M}_g$. We define its *curve operator* $\text{Add}_p(\gamma) \in \text{End}(V_p(\Sigma_g))$ as follows. Given $L \subset H_g$ a framed link, using $\alpha : \Sigma_g \cong \partial H_g$, we can draw $\alpha(L)$ in ∂H_g and then pushing inside H_g obtaining a framed link $L \cup \gamma$. The operator $\text{Add}_p(\gamma)$ is then defined by the assignation $[L] \rightarrow [L \cup \gamma]$ (see Figure 4).

Elementary surgery arguments (see e.g. [BHMV95]) imply that the set of curve operators generate $\text{End}(V_p(\Sigma_g))$. Note that it follows from their definition that the curve operators are self-adjoint for the invariant form.

An important property verified by the curve operators is the following *Egorov identity*:

$$\rho_p(\phi)^{-1} \text{Add}_p(\gamma) \rho_p(\phi) = \text{Add}_p(\phi(\gamma)), \quad \forall \gamma \in \mathcal{M}_g, \forall \phi \in \text{Mod}(\Sigma_g) \quad (2)$$

$$\text{Add} \left(\text{Diagram 1} \right) \cdot \text{Diagram 2} = \text{Diagram 3} \\ = - \left(A^4 + A^{-4} \right) \text{Diagram 4}$$

Figure 4: Action of a curve operator in genus one.

This identity is trivially satisfied when ϕ is one of the T_{x_i} 's or T_{z_i} 's which extends to H_g (see Figure 5). It also holds for the T_{y_i} 's by duality.

$$L \subset \Sigma_1$$

$$T \text{ Dehn Twist}$$

$$\rho(T) \text{Add}(L) e_i = \rho(T) \text{Diagram 1} = \text{Diagram 2}$$

$$\text{Add}(T \cdot L) \rho(T) e_i = \text{Add}(\text{Diagram 3}) \cdot \text{Diagram 4} = \text{Diagram 5}$$

Figure 5: Illustration of the Egorov identity in genus 1.

We can now prove that the $\rho_p(T_{x_i})$'s, the $\rho_p(T_{z_i})$'s and the $\rho_p(T_{y_i})$'s generate a projective representation. Let ϕ_1, \dots, ϕ_N be Humphries' generators that form a relation $R := \phi_1 \circ \dots \circ \phi_N = \mathbf{1}$. Write $\rho_p(R) := \rho_p(\phi_1) \circ \dots \circ \rho_p(\phi_N)$. To show we have a projective representation, we must show that $\rho_p(R)$ is scalar. By the Egorov identity (2), this operator commutes with all curves operators, thus commutes with the whole $\text{End}(V_p(\Sigma_g))$ and is scalar. Note that by composition, the Egorov identity indeed holds for any $\phi \in \text{Mod}(\Sigma_g)$.

For simplicity, we consider a central extension $\widetilde{\text{Mod}}(\Sigma_g)$ of $\text{Mod}(\Sigma_g)$ that lifts the above projective representations to linear ones (see [MR95, GM13] for explicit lifts) that we still denote ρ_p by abuse of notation:

$$\rho_p : \widetilde{\text{Mod}}(\Sigma_g) \rightarrow \text{U}(V_p(\Sigma_g))$$

These are the so-called (SU(2)) Reshetikhin-Turaev representations.

3 The quantum ergodic theorem

3.1 A geometric proposition

The proof of the main theorem relies on the following:

Proposition 3.1. .

Let (E, d) be a metric space and $\Delta \subset E$ be a convex compact subset.

For all $p \geq 0$, fix an integer $N_p > 0$ and some points $\tau_{1,p}, \dots, \tau_{N_p,p} \in \Delta$ together with weights $\alpha_{1,p}, \dots, \alpha_{N_p,p} \in [0, 1]$ such that $\sum_i \alpha_{i,p} = 1$.

Denote by $B_p := \sum_i \alpha_{i,p} \tau_{i,p}$ the barycenter of the weighted points. Eventually choose $\tau \in \Delta$ an extremal point of Δ .

Suppose that:

$$B_p \xrightarrow[p \rightarrow \infty]{d} \tau$$

Then there exist subsets $J_p \subset \{1, \dots, N_p\}$ such that:

1. If we note $\|J_p\| := \sum_{i \in J_p} \alpha_{i,p}$, then:

$$\|J_p\| \xrightarrow[p \rightarrow \infty]{} 1$$

2. For any sequence $j = (j_p)_{p \geq 1}$ with $j_p \in J_p$, then

$$\tau_{j_p,p} \xrightarrow[p \rightarrow \infty]{d} \tau$$

Figure 6 illustrates the Proposition by showing two sets of points inside a compact convex at two different moments. When the barycenter approaches an extremal point, then 'almost all' points approach it as well. This is our geometric interpretation of the Schnirelman theorem. Note that it might happen that a few points with small weight do not converge to the extremal point, such as the point $\tau_{i,p'}$ on the right picture. Such exceptional sequence are called *Scars* in the quantum chaotic literature.

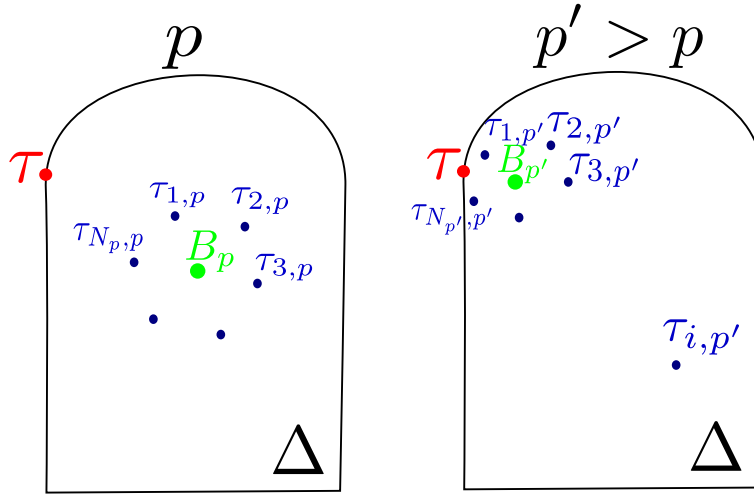


Figure 6: Illustration of Proposition 3.1.

The proof of Proposition 3.1 will be deduced from the following:

Lemma 3.2. Let (E, d) be a metric space, $\tau_1, \dots, \tau_N \in E$ be some points equipped with weights $\alpha_1, \dots, \alpha_N \in [0, 1]$ such that $\sum_i \alpha_i = 1$.

Denote by $B := \sum_i \alpha_i \tau_i$ their barycenter and by $\Delta := \text{ch}(\tau_1, \dots, \tau_N)$ their convex hull. For a subset $K \subset E$, we will use the notation $|K| := \sum_{\tau_i \in K} \alpha_i \in [0, 1]$.

The followings hold:

1. There exists a linear form $L \in E'$, in the dual E' of E , there exist constants $a, b \in \mathbb{R}$ so that $L(\Delta) = [a, b]$ and $L^{-1}(a) = \tau$.
2. Let $\epsilon > 0$ and $0 < \delta < 1$ and choose $c \in (a, b)$ such that:

$$F_c := \{x \in \Delta, \text{ s.t } L(x) \leq c\} \subset B(\tau, \epsilon)$$

where $B(\tau, \epsilon)$ denotes the ball of center τ and radius ϵ .

Noting $\alpha := |F_c|$, we have:

$$d(\tau, B) > \frac{(1 - \alpha)c - a}{\|L\|}$$

3. For all $\epsilon > 0$ and for all $0 < \delta < 1$, there exists $d > 0$ such that:

$$d(\tau, B) \leq d \implies |B(\tau, \epsilon)| \geq \delta$$

Remark. In this lemma, instead of having a general convex compact as in Proposition 3.1, we choose the convex hull of the points τ_i . The reason for it, is that for a general extremal point of a convex compact subset of a metric space, the first point of Lemma 3.2 is not always satisfied. An extremal point have point 1 satisfied is called an *exposed point* in literature. The point τ of Figure 6 is an exemple of a not exposed extremal point.

Proof of 3.2. It is a general fact (see e.g. [Bré11]) that in any locally compact space E , we can find a linear form $L \in E'$ separating finite sets of points, that is such that there exists $c \in \mathbb{R}$ such that:

$$L(\tau) < c < L(\tau_i) \quad , \text{ for all } i \in \{1, \dots, N\}$$

We obtain the first point by noting $a := L(\tau)$ and $b := \max_i(L(\tau_i))$.

To prove the second one, note the sub-barycenter $B_1 := \sum_{i|\tau_i \in F_c} \alpha_i$ and $B_2 := \sum_{i|\tau_i \notin F_c} \alpha_i$.

By convexity of $\Delta \setminus F_c$, we have $L(B_2) > c$. Moreover, since:

$$B = \alpha B_1 + (1 - \alpha) B_2$$

we have:

$$L(B) = \alpha L(B_1) + (1 - \alpha) L(B_2)$$

Using the fact that L is continuous, we obtain:

$$\begin{aligned} d(\tau, B) &\geq \frac{|L(B) - L(\tau)|}{\|L\|} = \frac{\alpha L(B_1) + (1 - \alpha) L(B_2) - a}{\|L\|} \\ &\geq \frac{(1 - \alpha) L(B_2) - a}{\|L\|} \\ &> \frac{(1 - \alpha)c - a}{\|L\|} \end{aligned}$$

which proves the second point.

To prove the last point, we remark that since $B(\tau, \epsilon) \subset F_c$, we have $|B(\tau, \epsilon)| \leq |F_c| = \alpha$. Thus:

$$d(\tau, B) > \frac{(1 - |B(\tau, \epsilon)|)c - a}{\|L\|}$$

So $d := \frac{(1 - \delta)c - a}{\|L\|}$ verifies the conclusion of the theorem. □

Proof of Proposition 3.1. .

Apply the third point of Lemma 3.2 to the polytope $\Delta_p := \text{ch} \{ \tau_{1,p}, \dots, \tau_{N_p,p}, \tau \}$ with $\epsilon = \frac{1}{p}$ and $\delta = 1 - \frac{1}{p}$. We obtain $d_p > 0$ such that:

$$d(\tau, B_p) \leq d_p \implies \left| B(\tau, \frac{1}{p}) \right| \geq 1 - \frac{1}{p}$$

Since $(B_p)_p$ converge to τ , there exists a rank $r(p)$ such that:

$$\begin{aligned} r \geq r(p) &\implies d(\tau, B_p) \leq d_p \\ &\implies \left| B(\tau, \frac{1}{p}) \right| \geq 1 - \frac{1}{p} \end{aligned}$$

We can suppose that the sequence $(r(p))_p$ is strictly increasing. If $r(p) \leq r < r(p+1)$, we define:

$$J_r := \left\{ j, \text{ s.t. } d(\tau, \tau_{j,r}) \leq \frac{1}{p} \right\}$$

1. Since $r \geq r(p)$, we have $|J_r| = \left| B(\tau, \frac{1}{p}) \right| \geq 1 - \frac{1}{p}$. Thus:

$$|J_r| \xrightarrow{r \rightarrow \infty} 1$$

2. If $j = (j_r)_r$ with $j_r \in J_r$, we have $d(\tau, \tau_{j_r,r}) \leq \frac{1}{p}$. Thus:

$$d(\tau, \tau_{j_r,r}) \xrightarrow{r \rightarrow \infty} 0$$

□

3.2 The quantum ergodic theorem

We now restate Theorem 1.1 in a slightly more general form. Let (\mathcal{X}, μ) be a compact Hausdorff space equipped with a measure on the borelian σ -algebra. Let G be a group acting on \mathcal{X} by preserving μ . Let

$$\rho_p : G \rightarrow \text{U}(V_p) \quad , \text{ for } p \geq p_0$$

be a family of representations on complex finite dimensional vector spaces V_p which preserve some non-degenerate positive definite form $\langle \cdot, \cdot \rangle_p$.

For each $p \geq p_0$, we fix a decomposition:

$$V_p = W_{1,p} \oplus \dots \oplus W_{N_p,p}$$

such that each $W_{i,p}$ is G -invariant.

Denote by $C(\mathcal{X})$ the algebra of continuous functions on \mathcal{X} and let $(f_n)_{n \geq 0}$ be a dense family for the norm $\| \cdot \|_\infty$. We suppose there exist onto morphisms:

$$\Psi_p : C(\mathcal{X}) \twoheadrightarrow \text{End}(V_p)$$

which are G -equivariant, where G acts on $\text{End}(V_p)$ by conjugacy. We also suppose that the operators $\Psi(f_n)$ are self-adjoint, that $\Psi(f_n^2) = \Psi(f_n)^2$ and that $\Psi(1) = \mathbb{1}$.

We define the following states on the algebra $C(\mathcal{X})$:

$$\tau(f) := \frac{1}{\text{Vol}(\mathcal{X})} \int_{\mathcal{X}} f d\mu$$

and:

$$B_p(f) := \frac{1}{\dim(V_p)} \text{Tr}(\Psi(f))$$

Also, denoting by $P_{i,p} \in \text{End } V_p$ the orthogonal projector on $W_{i,p}$, we define:

$$\tau_{i,p}(f) := \frac{1}{\dim(W_{i,p})} \text{Tr}(P_{i,p}\Psi(f))$$

The assumptions on Ψ imply that B_p and the $\tau_{i,p}$ are states on the $*$ -algebra $C(\mathcal{X})$, that is, that they are linear forms verifying $\tau(f) \geq 0$ if $f \geq 0$ and $\tau(1) = 1$. It follows from the Riesz theorem that they correspond to probability measures on \mathcal{X} (equipped with the borelian σ -algebra) and they are obviously G -invariants. Note also that if we note $\alpha_{i,p} := \frac{\dim(W_{i,p})}{\dim(V_p)}$, then we have $\sum_i \alpha_{i,p} = 1$ and $B_p = \sum_i \alpha_{i,p} \tau_{i,p}$. We can now state the main theorem of this paper:

Theorem 3.3. .

With the preceding notations, suppose that:

- *The action of G on (\mathcal{X}, μ) is ergodic.*
- *For all $f \in C(\mathcal{X})$, we have:*

$$B_p(f) \xrightarrow[p \rightarrow \infty]{} \tau(f)$$

Then for all $p \geq p_0$, there exists a subset $J_p \subset \{1, \dots, N_p\}$ such that:

1. *We have:*

$$\sum_{i \in J_p} \alpha_{i,p} \xrightarrow[p \rightarrow \infty]{} 1$$

2. *For any sequence $j = (j_p)_p$ with $j_p \in J_p$, we have:*

$$\tau_{j_p,p}(f) \xrightarrow[p \rightarrow \infty]{} \tau(f) \quad , \text{ for all } f \in C(\mathcal{X})$$

Remark. 1. Theorem 1.1, the $SU(2)$ version, is easily deduced from this one when choosing $\mathcal{X} = \mathcal{X}_g$, $V_p = V_p(\Sigma_g)$ and $f_n = f_{\gamma_n}$ where $(\gamma_n)_n$ is any numbering of the countable set \mathcal{M}_g . The morphism Ψ is defined by $\Psi_p(f_\gamma) = \text{Add}_p(\gamma)$. The second assumption in the theorem is Marché-Narrimanejad theorem (1) from [MN08].

2. This theorem is a generalization of the main theorem of Bouzouina-de Bièvre [BDB96], which in turn is a variant of the Schnirelman theorem from [Sch74] (see also [Zel87, CdV85, HMR87] for different variants of this theorem). We recover the theorem from [BDB96] when we choose for \mathcal{X} the two dimensional torus \mathbb{T} equipped with its natural symplectic measure, for V_p the Segal-Shale-Weil representations which factorize through the symplectic group, and when $G = \mathbb{Z}$ where 1 acts on the torus through an Anosov element. In this case, the authors decompose the spaces V_p into invariant one dimensional eigenspaces and the conclusion is the same than ours. Note that their proof, and those of all other variations, heavily relies on the Birkhoff theorem, which is known only for a restricted class of groups, like the groups with polynomial growth or the hyperbolic ones, but not for the mapping class groups or the Torelli groups. Thus their proofs can not be translated in our general context.
3. Still it might happen that a few sequences $(\tau_{i_p,p})_p$ do not converge to τ if the associate dimensions are not too large, as illustrated in Figure 6. Such exceptional sequences are called *Scars* in the quantum chaotic literature. Such Scars have been found in the abelian case in [FNDB03, Kel07]. The next Corollary provide a condition for not being a Scar.

We immediately deduce from Theorem 3.3 the following:

Corollary 3.4. *Under the assumptions of Theorem 3.3, if $(j_p)_p$ is an exceptional sequence such that $(\tau_{j_p,p})_p$ do not converge toward τ (in the $*$ -weak topology), then we must have:*

$$\frac{\dim(W_{j_p,p})}{\dim(V_p)} \xrightarrow{p \rightarrow \infty} 0$$

Proof of Theorem 3.3. The theorem will be deduced from Proposition 3.1.

Choose for (E, d) the dual of $C(\mathcal{X})$ for the norm $\|f\|_\infty = \sup_{x \in \mathcal{X}} \|f(x)\|$, equipped with the distance:

$$d(f, g) := \sum_{n \geq 1} \frac{1}{2^n} d(f_n, g_n)$$

The induced topology is the so-called $*$ -weak topology and a sequence $(\tau_n)_n$ converge towards τ for d if and only if:

$$\tau_n(f) \xrightarrow{n \rightarrow \infty} \tau(f) \quad , \text{ for all } f \in C(\mathcal{X})$$

We refer to [Br  11] for detailed discussions on this metric space.

The set of states on $(C(\mathcal{X}), \|\cdot\|_\infty)$ is in bijection with the set of probability measures on \mathcal{X} equipped with the Borelian σ -algebra. The set of states invariants for the action of G is compact (because closed in the compact unit ball) and convex. We denote it by Δ . Its extremal points correspond precisely to the measures for which the action of G is ergodic. Thus the first hypothesis of the theorem implies that τ is an extremal point of Δ .

As we said previously, B_p is the barycenter of the point $\tau_{i,p}$ for the weights $\alpha_{i,p} := \frac{\dim(W_{i,p})}{\dim(V_p)}$, thus the second hypothesis of the theorem states that the sequence of barycenters converge to τ .

We can thus apply Proposition 3.1 to conclude. \square

4 Application to the asymptotic of the spin-decompositions

The goal of this section is to prove Corollary 1.2. We first briefly recall from [BHMV95] the definition of the spin-decompositions for self-completeness of the paper.

Choose $p \geq 8$ such that 4 divides p and consider the group $G_p \subset \text{Mod}(\Sigma_g)$ generated by the p -th power of Dehn twists. In [BHMV95] Proposition 7.5 and Remark 7.6, it is shown that the image $\rho_p(G_p)$ is isomorphic to the group \mathcal{H} defined by:

$$\mathcal{H} := \begin{cases} H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}) & , \text{ if } p \equiv 4 \pmod{8} \\ H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z} & , \text{ if } p \equiv 0 \pmod{8} \end{cases}$$

The isomorphism being given by the natural map sending $\rho_p(T_\gamma)^p$ to the class $[\gamma] \in \mathcal{H}$. We thus obtain a decomposition:

$$V_p(\Sigma_g) = \bigoplus_{\chi \in \text{Hom}(\mathcal{H}, \mathbb{Z}/2\mathbb{Z})} V_p(\Sigma_g, \chi)$$

where:

$$V_p(\Sigma_g, \chi) := \{v \in V_p(\Sigma_g), \text{ s.t. } \rho_p(T_\gamma)^p v = \chi([\gamma])v\}$$

It follows from the Egorov identity (2) of section 2, that $V_p(\Sigma_g, \chi = 0)$ is preserved by $\text{Mod}(\Sigma_g)$ and that all the $V_p(\Sigma_g, \chi)$ are preserved by the Torelli group.

Note that, when 8 divides p , then $\text{Hom}(\mathcal{H}, \mathbb{Z}/2\mathbb{Z})$ is in bijection with the set of spin-structures of Σ_g , thus justifying the name *spin-decomposition*. We refer also to [Mar11] for a geometric interpretation of these decompositions.

To prove Corollary 1.2 we want to apply Corollary 3.4. We thus need to control the dimensions of the spaces $V_p(\Sigma_g, \chi)$. These dimensions were computed in [BHMV95] from which we deduce the following, from which the proof of Corollary 1.2 is immediate.

Lemma 4.1. *We have the following:*

1.

$$\dim(V_{4r}(\Sigma_g)) \underset{r \rightarrow \infty}{\sim} \text{Vol}(\mathcal{X}_g)(4r)^{3g-3}$$

2.

$$\dim(V_{4r}(\Sigma_g, \chi)) \underset{r \rightarrow \infty}{\sim} 2^{-2g} \text{Vol}(\mathcal{X}_g)(4r)^{3g-3}, \text{ for all } \chi \in \mathcal{H}$$

Proof. The computation of $\dim(V_p(\Sigma_g))$ was first done by Verlinde in [Ver88] under the framework of the Wess-Zumino-Witten conformal field theory. An alternative elementary computation was done in [BHMV95] Corollary 1.16. In particular, it follows from these formulas that $\dim(V_p(\Sigma_g))$ is a polynomial in p of degree $3g - 3$. Using Marché-Narimanejad theorem (1) of [MN08] with f the constant function equal to one, we deduce the first equivalence.

In [BHMV95] Theorem 7.10 and 7.16, the authors computed $\dim(V_{4r}(\Sigma_g, \chi))$. When r is odd, they showed that:

$$\dim(V_{4r}(\Sigma_g, \chi \neq 0)) = 2^{-2g} (\dim(V_{4r}(\Sigma_g)) - r^{g-1})$$

and:

$$\dim(V_{4r}(\Sigma_g, \chi = 0)) = \dim(V_{4r}(\Sigma_g, \chi \neq 0)) + r^{g-1}$$

However when r is even, denoting by $\epsilon \in \{0, 1\}$ the Arf invariant of χ , they showed the formula:

$$\dim(V_{4r}(\Sigma_g, \chi)) = 2^{-2g} (\dim(V_{4r}(\Sigma_g)) + r^{g-1} ((-1)^\epsilon 2^g - 1))$$

In particular, in every cases, $\dim(V_{4r}(\Sigma_g, \chi))$ is a polynomial in r of leading term $2^{-2g} \text{Vol}(\mathcal{X}_g)(4r)^{3g-3}$. \square

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